Grading guide, Pricing Financial Assets, June 2019

1. In Vasicek's model, the risk-neutral process for the short rate r is

$$dr = a(b - r)\,dt + \sigma dz$$

where a, b, σ are constants. Suppose $a > 0$.

   (a) Describe the qualitative characteristics of this model of the short rate.

   (b) Consider a zero-coupon bond that pays \$1 at time T. Vasicek shows that its price at time t, denoted $P(t,T)$, can be expressed

$$P(t,T) = A(t,T)\,e^{-B(t,T)r(t)}$$

   using functions A and B that he finds (which we don't need in detail right here). Show that

$$\frac{\partial P(t,T)}{\partial r(t)} = -B(t,T)\,P(t,T).$$

   (c) Argue that the process for $P(t,T)$ in the traditional risk-neutral world satisfies

$$dP(t,T) = r(t)\,P(t,T)\,dt - \sigma B(t,T)\,P(t,T)\,dz(t).$$

Solution:

(a) Cf. Hull p. 682–684. The model incorporates mean reversion, as the short rate is pulled to level b at rate a. Page 684 explains why mean reversion is desirable for a model of the short rate. Superimposed upon this mean reversion is a normally distributed stochastic term σdz.

(b) The first expression appears on page 685. The derivative of $P(t,T)$ with respect to $r(t)$ becomes

$$\frac{\partial P(t,T)}{\partial r(t)} = A(t,T)\left(-B(t,T)\,e^{-B(t,T)r(t)}\right) = -B(t,T)\,P(t,T).$$

This is also (30.9).

(c) Cf. Hull p. 687. The expected growth rate of $P(t,T)$ in the traditional risk-neutral world is $r(t)$ because $P(t,T)$ is the price of a traded security. The coefficient of $dz(t)$ in the process for $P(t,T)$ can be calculated from Itô's lemma as $\sigma\partial P(t,T)/\partial r(t)$. The claimed process for $P(t,T)$ follows from these considerations, using the answer from (b).

2. Suppose f and g are the prices of two non-dividend paying securities with volatilities σ_f and σ_g, depending on a single source of risk captured by the Wiener process z. Suppose that the market price of risk is σ_g. Let r denote the risk-free rate.

   (a) Argue that f and g satisfy

$$df = (r + \sigma_g\sigma_f)\,f dt + \sigma_f f dz \text{ and } dg = \left(r + \sigma_g^2\right) g dt + \sigma_g g dz.$$

(b) Apply Itô's lemma to find $d\ln f$ and $d\ln g$. Recall that $\ln\left(\frac{f}{g}\right) = \ln f - \ln g$. Show that

$$d\left(\ln\frac{f}{g}\right) = -\frac{(\sigma_f - \sigma_g)^2}{2}dt + (\sigma_f - \sigma_g)\,dz.$$

(c) Using the equation for $\ln\left(\frac{f}{g}\right)$ from (b), apply Itô's lemma to determine the equation for f/g. Show that f/g has zero drift, and comment on the interpretation of this result.

Solution:

(a) From Hull 27.1, when λ is the market price of the single source of risk, security price f obeys

$$df = (r + \lambda\sigma_f)\,f dt + \sigma_f f dz. \qquad (27.10)$$

Once $\lambda = \sigma_g$, the two expressions follow.

(b) Cf. Hull p. 636. First, Itô's lemma gives

$$d\ln f = \left(r + \sigma_g\sigma_f - \frac{\sigma_f^2}{2}\right)dt + \sigma_f dz$$

and, likewise,

$$d\ln g = \left(r + \frac{\sigma_g^2}{2}\right)dt + \sigma_g dz.$$

Then

$$d\ln f - d\ln g = \left(\sigma_g\sigma_f - \frac{\sigma_f^2}{2} - \frac{\sigma_g^2}{2}\right)dt + (\sigma_f - \sigma_g)\,dz,$$

which is identical to the desired expression.

(c) If we let $x = \ln\left(\frac{f}{g}\right)$, we're seeking to find the equation for $G(x) = e^x = \frac{f}{g}$. Itô's lemma gives

$$d\frac{f}{g} = (\sigma_f - \sigma_g)\,\frac{f}{g}dz,$$

which indeed has zero drift. To focus on the drift term, from (13.12) it is

$$\frac{\partial G}{\partial x}\left(-\frac{(\sigma_f - \sigma_g)^2}{2}\right) + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\sigma_f - \sigma_g)^2 = 0$$

since the derivative of e^x is also e^x. The interpretation is that the relative security price $\frac{f}{g}$ is a martingale. As explained on the top of p. 636, this is the equivalent martingale measure result.

3. Consider a futures contract on a non-dividend paying stock with a time to maturity of T. Let the current $(t = 0)$ price of the stock be S_0, and assume that the risk-free rate of interest is a constant r.

(a) The futures price F_0 at $t = 0$ satisfies $F_0 = S_0 e^{rT}$. What is the delta of the futures contract?

(b) Compare the price of a European call option on the stock with the price of a European call option on the futures contract when the two options have the same strikes and maturities, and they mature at the same time as the futures contract.

(c) Explain the following in words (no mathematical derivation is expected): in a risk-neutral world, the futures price behaves in the same way as the price of a stock that pays dividend yield r.

Solution:

(a) Cf. Hull p. 399. The delta is

$$\frac{\partial F_0}{\partial S_0} = e^{rT}.$$

(b) Following Hull 17.3, towards maturity when T is close to zero, the futures price approximates the stock price. These two options therefore have identical payoffs at expiry. Being European, then they have identical values also before expiry.

(c) Cf. Hull 17.7 for full explanations. Entering into a long futures contract has value zero. Therefore, under risk-neutral valuation, $F_0 = \hat{E}[F_T]$ for any time T. The exact same martingale property, $S_0 = \hat{E}[S_T]$, holds or a stock that pays dividend at rate r. Its drift is $r - r = 0$.